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$\mathbb{Z}_2$-gradings of Clifford algebras and multivector structures

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Abstract
Let $\mathcal{C}\ell(V, g)$ be the real Clifford algebra associated with the real vector space V, endowed with a nondegenerate metric g. In this paper, we study the class of $\mathbb{Z}_2$-gradings of $\mathcal{C}\ell(V, g)$ which are somehow compatible with the multivector structure of the Grassmann algebra over V. A complete characterization for such $\mathbb{Z}_2$-gradings is obtained by classifying all the even subalgebras coming from them. An expression relating such subalgebras to the usual even part of $\mathcal{C}\ell(V, g)$ is also obtained. Finally, we employ this framework to define spinor spaces, and to parametrize all the possible signature changes on $\mathcal{C}\ell(V, g)$ by $\mathbb{Z}_2$-gradings of this algebra.

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1. Introduction

Clifford algebras have long been an important tool in the interplay among geometry, algebra and physics. The development of the theory of spinor structures, with applications in field and string theories, and the study of Dirac operators, with applications in geometry and topology, are examples of this general setting. These algebras carry $\mathbb{Z}_2$-graded structures which play a major role in such developments. For example, the usual $\mathbb{Z}_2$-graded structure of the Clifford bundle over a Riemannian manifold may be used to construct models of supersymmetric quantum mechanics, which have unveiled deep connections between field theory and geometry

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[1]. Also, in Hestenes' approach to Dirac theory [2–4], the usual $\mathbb{Z}_2$-grading of the spacetime algebra is extensively employed to represent spinors by even elements of this algebra (such an approach has a natural generalization for arbitrary Clifford algebras [5]).

Let V be a finite-dimensional real vector space endowed with a metric g. By this we mean that $g: V \times V \to \mathbb{R}$ is a bilinear, symmetric and nondegenerate map. Let $\mathcal{C}\ell(V, g)$ be the real Clifford algebra associated with (V, g). As a *vector space*, $\mathcal{C}\ell(V, g)$ is naturally $\mathbb{Z}$-graded by the multivector structure inherited from the Grassmann algebra $\Lambda(V)$ over V. This is the usual Chevalley construction (see equation (2)). However, as an *algebra*, $\mathcal{C}\ell(V, g)$ is not $\mathbb{Z}$-graded, but only $\mathbb{Z}_2$-graded and, in general, such $\mathbb{Z}_2$-gradings do not have to preserve in any sense the homogeneous subspaces of $\Lambda(V) \cong \mathcal{C}\ell(V, g)$ ($\cong$ denotes *linear* isomorphism in this expression).

In this paper, we study the class of $\mathbb{Z}_2$-gradings of $\mathcal{C}\ell(V, g)$ which are somehow compatible with the usual multivector structure of $\Lambda(V)$ (see definition 3). In section 2, we completely characterize such $\mathbb{Z}_2$-gradings by classifying all the even subalgebras coming from them. Also, a formula relating such arbitrary even subalgebras to the usual even part of $\mathcal{C}\ell(V, g)$ is obtained.

In the next section, some preliminary applications are considered. We start by discussing the possibility of employing these arbitrary $\mathbb{Z}_2$-gradings to define spinor spaces, as in [5, 6]. After that, we consider the problem of signature change in an arbitrary Clifford algebra. There are various situations in theoretical physics where changing the signature of a given space is an useful tool, as in Euclidean formulations of field theories, in the theory of instantons, in finite temperature field theory and in lattice gauge theory. In [7, 8], the authors discuss the specific signature changes $(1, 3) \to (3, 1)$ and $(1, 3) \to (4, 0)$ inside the spacetime algebra (in the last case, the corresponding signature change map is used to study the Dirac equation, and self-dual/anti-self-dual solutions of gauge fields). In section 3.2, we generalize such approaches in order to obtain completely arbitrary signature change maps in Clifford algebras of any dimension. As in the aforementioned works, our method is *purely algebraic*, and is implemented by a deformation of the algebraic structure underlying the theory. More specifically, the $\mathbb{Z}_2$-gradings discussed above are employed to deform the original Clifford product, thereby 'simulating' the product properties of the signature-changed space. As a result, we parametrize all the possible signature changes on $\mathcal{C}\ell(V, g)$ by $\mathbb{Z}_2$-gradings of this algebra. This opens the possibility of applying this formalism to higher dimensional physical theories.

The concept of $\mathbb{Z}_2$-graded structures has numerous applications in mathematical physics (as in supersymmetry, supergeometry, etc). It is then reasonable to expect that the present work may find other applications besides those considered here and outlined above.

1.1. Algebraic preliminaries and notation

A vector space W is said to be graded by an Abelian group G if it is expressible as a direct sum $W = \bigoplus_i W_i$ of subspaces labelled by elements $i \in G$ (we refer the reader to appendix A of [9] for a general review of algebraic concepts). Here we consider only the cases when G is given by $\mathbb{Z}$ or $\mathbb{Z}_2$. Then, the elements of W_i are called homogeneous of degree i and we define $\deg(w) = i$ if $w \in W_i$. Let $\mathcal{A}$ be an algebra which, for the purposes of this paper, can always be considered as a finite-dimensional associative algebra with unit, over $\mathbb{R}$ or $\mathbb{C}$. We say that $\mathcal{A}$ is graded by G if (a) its underlying vector space is a G-graded vector space and (b) its product satisfies $\deg(ab) = \deg(a) + \deg(b)$.

Let V be an n-dimensional real vector space. Then, the tensor algebra $T(V) = \bigoplus_{k=0}^{\infty} T^k(V)$ over V is an example of a $\mathbb{Z}$-graded algebra. We denote the space of antisymmetric k-tensors by $\Lambda^k(V)$. The elements of this space will be called k-vectors. Let $V^\wedge = \bigoplus_{k=0}^n \Lambda^k(V)$ denote the 2^n-dimensional real vector space of multivectors over V. Using the natural embeddings of $\mathbb{R}$ and V in $V^\wedge$, we identify $\Lambda^0(V)$ with $\mathbb{R}$ and $\Lambda^1(V)$ with V. When endowed with the exterior product $\wedge$, the vector space $V^\wedge$ becomes the so-called Grassmann algebra $\Lambda(V) = (V^\wedge, \wedge)$ over V. We note that $\Lambda(V) = \bigoplus_{k=0}^n \Lambda^k(V)$ is another example of a $\mathbb{Z}$-graded algebra, with a $\mathbb{Z}$-graded structure inherited from the usual $\mathbb{Z}$-grading of $T(V)$. It is important to note that such $\mathbb{Z}$-grading for $\Lambda(V)$ is by no means unique [10, 11]. Nevertheless, suppose that one wants to identify V with the tangent space (at a certain point) of a spacetime M. Then, in the context of this usual grading, one can interpret elements of $\Lambda^0(V)$ as scalars, elements of $\Lambda^1(V)$ as tangent vectors of M and so on. In this paper, we always consider the multivector structure coming from such usual $\mathbb{Z}$-grading of $\Lambda(V)$ (more discussion along these lines can be found in [6]).

We denote the projection of a multivector $a = a_0 + a_1 + \cdots + a_n$, with $a_k \in \Lambda^k(V)$, on its p-vector part by $\langle a\rangle_p := a_p$. The parity operator $(\cdot)^\wedge$ is defined as the algebra automorphism generated by the expression $\hat{v} = -v$ on vectors $v \in V$. The reversion $(\cdot)^\sim$ is the algebra anti-automorphism generated by the expression $\tilde{v} = v$ on vectors $v \in V$. It follows that $\hat{a} = (-1)^k a$ and $\tilde{a} = (-1)^{[k/2]} a$ if $a \in \Lambda^k(V)$, where $[m]$ denotes the integer part of m. When V is endowed with a metric g, it is possible to extend (in a non-unique way) g to all of $V^\wedge$. Given $a = u_1 \wedge \cdots \wedge u_k$ and $b = v_1 \wedge \cdots \wedge v_l$ with $u_i, v_j \in V$, the expressions $g(a, b) = \det(g(u_i, v_j))$, if $k = l$, and $g(a, b) = 0$, if $k \neq l$, provide one such extension. Also, the left ($\lrcorner$) and right ($\llcorner$) contractions on the Grassmann algebra are respectively defined by $g(a \lrcorner b, c) = g(b, \tilde{a} \wedge c)$ and $g(b \llcorner a, c) = g(b, c \wedge \tilde{a})$, with $a, b, c \in \Lambda(V)$.

The Clifford product between a vector $v \in V$ and a multivector a in $V^\wedge$ is given by $va = v \wedge a + v \lrcorner a$. This is extended by linearity and associativity to all of $V^\wedge$. The resulting algebra is the so-called Clifford algebra $\mathcal{C}\ell(V, g)$. Note that, although the underlying vector space of $\mathcal{C}\ell(V, g)$ (i.e. $V^\wedge$) is $\mathbb{Z}$-graded, $\mathcal{C}\ell(V, g)$ is not a $\mathbb{Z}$-graded algebra as, for example, the Clifford product between two 1-vectors is a sum of elements of degrees 0 and 2. Nevertheless, there are (infinite) $\mathbb{Z}_2$-gradings which are compatible with the Clifford product structure. For instance, the usual $\mathbb{Z}_2$-grading of $\mathcal{C}\ell(V, g)$ is given by $\mathcal{C}\ell^+(V, g) \oplus \mathcal{C}\ell^-(V, g)$ where $\mathcal{C}\ell^+(V, g) = \bigoplus_{k\,\text{even}} \Lambda^k(V)$ and $\mathcal{C}\ell^-(V, g) = \bigoplus_{k\,\text{odd}} \Lambda^k(V)$. When the metric g has signature (p, q), we will also denote the real vector space V endowed with g by $\mathbb{R}^{p,q}$. In this case, the real Clifford algebra $\mathcal{C}\ell(V, g)$ over V will be denoted by $\mathcal{C}\ell_{p,q}(\mathbb{R})$ or $\mathcal{C}\ell_{p,q}$. We adopt the definition $\mathcal{C}\ell_{p,q}(\mathbb{C}) = \mathcal{C}\ell_{p,q}(\mathbb{R}) \otimes \mathbb{C}$ for the complexified Clifford algebra (of course, all the $\mathcal{C}\ell_{p,q}(\mathbb{C})$ with fixed $p + q$ are isomorphic as complex algebras). Note that given 1-vectors $x, y \in \mathbb{R}^{p,q}$, we have $2g(x, y) = xy + yx$. In particular, an orthonormal basis $\{e_i\}$ of $\mathbb{R}^{p,q}$ yields $e_i e_j + e_j e_i = 2g_{ij}$, where $g_{ij} = g(e_i, e_j)$. In the following, we denote by $\mathcal{M}(m, \mathbb{K})$ the space of $m \times m$ matrices over $\mathbb{K}$, where $\mathbb{K} = \mathbb{R}, \mathbb{C}$ or $\mathbb{H}$.

We observe that there are other ways of defining Clifford and Grassmann algebras (see, for example, chapter 14 of [7] and chapters 1 and 2 of [9]). In the definitions adopted here, both the Grassmann and the Clifford algebras are defined on the same underlying vector space $V^\wedge$. This will be particularly useful in section 3.2, where we consider various Clifford products defined, at the same time, on $V^\wedge$.

It is well known that real Clifford algebras exhibit an 8-fold periodicity and can be classified by $\mathcal{C}\ell_{p,q}(\mathbb{R}) \cong \mathcal{M}(m, \mathbb{R}) \otimes \mathcal{A}$, where $\mathcal{A}$ is given by table 1 and m is fixed by $m^2 \dim_{\mathbb{R}} \mathcal{A} = 2^n$, with $n = p + q$.

The usual even subalgebras $\mathcal{C}\ell^+_{p,q}(\mathbb{R})$ can be shown to satisfy

$$\mathcal{C}\ell^+_{p,q}(\mathbb{R}) \cong \mathcal{C}\ell_{q,p-1}(\mathbb{R}) \cong \mathcal{C}\ell_{p,q-1}(\mathbb{R}) \cong \mathcal{C}\ell^+_{q,p}(\mathbb{R}). \tag{1}$$

In this way, their classification follows from table 1, as table 2 shows.

Table 1. Classification of real Clifford algebras $\mathcal{C}\ell_{p,q}(\mathbb{R}) \cong \mathcal{M}(m,\mathbb{R}) \otimes \mathcal{A}$, where $m^2 \dim_{\mathbb{R}} \mathcal{A} = 2^n$ and $n = p+q$.

$p-q$ (mod 8)	0	1	2	3	4	5	6	7
$\mathcal{A}$	$\mathbb{R}$	$\mathbb{R}\oplus\mathbb{R}$	$\mathbb{R}$	$\mathbb{C}$	$\mathbb{H}$	$\mathbb{H}\oplus\mathbb{H}$	$\mathbb{H}$	$\mathbb{C}$

Table 2. Usual even parts of real Clifford algebras $\mathcal{C}\ell^+_{p,q}(\mathbb{R}) \cong \mathcal{M}(m,\mathbb{R}) \otimes \mathcal{B}$, where $m^2 \dim_{\mathbb{R}} \mathcal{B} = 2^{n-1}$ and $n = p+q$.

$p-q$ (mod 8)	0	1	2	3	4	5	6	7
$\mathcal{B}$	$\mathbb{R}\oplus\mathbb{R}$	$\mathbb{R}$	$\mathbb{C}$	$\mathbb{H}$	$\mathbb{H}\oplus\mathbb{H}$	$\mathbb{H}$	$\mathbb{C}$	$\mathbb{R}$

Table 3. Classification of complex Clifford algebras.

Even dimension	$\mathcal{C}\ell_{2k}(\mathbb{C}) \cong \mathcal{M}(2^k,\mathbb{C})$
Odd dimension	$\mathcal{C}\ell_{2k+1}(\mathbb{C}) \cong \mathcal{M}(2^k,\mathbb{C}) \oplus \mathcal{M}(2^k,\mathbb{C})$

The classification of the complex Clifford algebras is simpler, as table 3 shows (in this table, $\cong$ denotes isomorphism of complex algebras).

Clifford algebras may also be characterized by their universal property, in the sense of the well-known theorem below.

Theorem 1. *Let V be a finite-dimensional real vector space endowed with a nondegenerate metric g. Let $\mathcal{A}$ be a real associative algebra with unit $1_{\mathcal{A}}$. Given a linear map $\Gamma : V \to \mathcal{A}$ such that $(\Gamma(v))^2 = g(v,v)1_{\mathcal{A}}$, there exists a unique homomorphism $\bar{\Gamma} : \mathcal{C}\ell(V,g) \to \mathcal{A}$ such that $\bar{\Gamma}|_V = \Gamma$.*

A map Γ as in theorem 1 will be called a *Clifford map* for the pair (V, g). An important example is given by the Clifford map $\Gamma : V \to \mathrm{End}(V^\wedge)$, defined by

$$\Gamma(v) = v \wedge + v \lrcorner \qquad (2)$$

which implements the well-known Chevalley identification of $\mathcal{C}\ell(V,g)$ with a subalgebra of $\mathrm{End}(V^\wedge)$.

2. $\mathbb{Z}_2$-gradings of Clifford algebras

By abuse of notation, we will denote an arbitrary $\mathbb{Z}_2$-grading of $\mathcal{C}\ell(V,g)$ simply by $\mathcal{C}\ell(V,g) = \mathcal{C}\ell_0 \oplus \mathcal{C}\ell_1$. In this way, (the vector space structure of) $\mathcal{C}\ell(V,g)$ is given by a direct sum of subspaces $\mathcal{C}\ell_i$, $i = 0, 1$, which satisfy

$$\mathcal{C}\ell_i\mathcal{C}\ell_j \subseteq \mathcal{C}\ell_{i+j\,(\mathrm{mod}\ 2)}. \qquad (3)$$

Of course, $\mathcal{C}\ell_0$ is then a subalgebra of $\mathcal{C}\ell(V,g)$. For each such decomposition we have an associated vector space automorphism $\alpha : \mathcal{C}\ell(V,g) \to \mathcal{C}\ell(V,g)$ defined by $\alpha|_{\mathcal{C}\ell_i} = (-1)^i\, \mathrm{id}_{\mathcal{C}\ell_i}$ (where id_W denotes the identity map on the space W). The projections π_i on $\mathcal{C}\ell_i$ are given by $\pi_i(a) = \frac{a+(-1)^i\alpha(a)}{2}$. We also denote $\pi_i(a) = a_i$. Note that α is an algebra isomorphism, for given $a, b \in \mathcal{C}\ell(V,g)$, we have $\alpha(ab) = \alpha\left(\sum_{ij} a_i b_j\right) = \sum_{ij}\alpha(a_i b_j) = \sum_{ij}(-1)^{i+j}a_i b_j = \sum_i(-1)^i a_i \sum_j(-1)^j b_j = \alpha(a)\alpha(b)$. For the usual $\mathbb{Z}_2$-grading, where

$\mathcal{C}\ell_0 = \mathcal{C}\ell^+_{p,q}(\mathbb{R})$ and $\mathcal{C}\ell_1 = \mathcal{C}\ell^-_{p,q}(\mathbb{R})$, the grading automorphism is simply given by $(\cdot)^\wedge$ (see section 1.1).

Given a $\mathbb{Z}_2$-grading of $\mathcal{C}\ell(V, g)$ as above, we refer to $\mathcal{C}\ell_0$ and $\mathcal{C}\ell_1$ as the *α-even* and *α-odd* parts of $\mathcal{C}\ell(V, g)$. Also, an element belonging to $\mathcal{C}\ell_0$ ($\mathcal{C}\ell_1$) will be called α-even (α-odd).

We observe that the scalar $1 \in \Lambda^0(V)$ is always α-even. Indeed, let us write $1 = e + o$, where $e = \pi_0(1)$ and $o = \pi_1(1)$. Left-multiplying this equation by e yields $e = e^2 + eo$. As e and e^2 are α-even and eo is α-odd, we must have $eo = 0$. Then, right-multiplying $1 = e + o$ by o yields $o = o^2$. As o is α-odd and o^2 is α-even, we thus have $o = 0$.

Let us now address the central point of the present paper. In the general $\mathbb{Z}_2$-gradings introduced so far, the even and odd projections do not have to preserve the multivector structure of $\Lambda(V)$ (see an example later). In other words, it is possible that the even or odd part of a k-vector comprises an inhomogeneous combination of elements of different degrees.

Proposition 2. *Let $\mathcal{C}\ell(V, g) = \mathcal{C}\ell_0 \oplus \mathcal{C}\ell_1$ be a $\mathbb{Z}_2$-grading with grading automorphism α. The following are equivalent*[5]*:*

(i) The projections π_i, $i = 0, 1$, preserve each $\Lambda^k(V)$, $k = 1, \ldots, n$;
(ii) $\pi_i(V) \subseteq V$, $i = 0, 1$;
(iii) α preserves each $\Lambda^k(V)$, $k = 1, \ldots, n$;
(iv) $\alpha(V) \subseteq V$.

Proof. It follows from the definition of π_i that $\pi_i(\Lambda^k(V)) \subseteq \Lambda^k(V)$ if, and only if, $\alpha(\Lambda^k(V)) \subseteq \Lambda^k(V)$. Thus (i) $\Leftrightarrow$ (iii) and (ii) $\Leftrightarrow$ (iv). Of course, (iii) $\Rightarrow$ (iv). Conversely, if we assume (iv), then $\alpha|_V$ (see footnote 5) is an isometry. Indeed, given $x, y \in V, 2g(\alpha(x), \alpha(y)) = \alpha(x)\alpha(y) + \alpha(y)\alpha(x) = \alpha(xy + yx) = 2g(x, y)$, since α is an algebra isomorphism and $\alpha(1) = 1$ (as we mentioned earlier). If $\{e_i\}$ is an orthonormal basis of V, every element $a \in \Lambda^k(V)$ can be written as a linear combination of terms such as $e_{i_1} \wedge \cdots \wedge e_{i_k} = e_{i_1} \cdots e_{i_k}$. As $\alpha|_V$ is an isometry, $\{\alpha(e_i)\}$ is also an orthonormal basis of V and thus $\alpha\big(e_{i_1} \cdots e_{i_k}\big) = \alpha\big(e_{i_1}\big) \cdots \alpha\big(e_{i_k}\big) = \alpha\big(e_{i_1}\big) \wedge \cdots \wedge \alpha\big(e_{i_k}\big) \in \Lambda^k(V)$. It follows that $\alpha(a) \in \Lambda^k(V)$, establishing (iv) $\Rightarrow$ (iii). □

Definition 3. *A $\mathbb{Z}_2$-grading fulfilling one (and hence all) of the conditions above will be said to preserve the multivector structure of $\Lambda(V)$.*

For this class of $\mathbb{Z}_2$-gradings, we have the following proposition.

Proposition 4. *Let $\mathcal{C}\ell(V, g) = \mathcal{C}\ell_0 \oplus \mathcal{C}\ell_1$ be a $\mathbb{Z}_2$-grading preserving the multivector structure of $\Lambda(V)$ and define $V_i := V \cap \mathcal{C}\ell_i$, $i = 0, 1$, i.e., V_0 (V_1) is the space of α-even (α-odd) 1-vectors (see footnote 5). Then $V = V_0 \oplus V_1$, with $V_0 = V_1{}^\perp$ (and $V_1 = V_0{}^\perp$). It follows that each subspace V_i is nondegenerate (i.e. g restricted to V_i is nondegenerate).*

Proof. By assumption, each projection π_i preserves V. This immediately induces a $\mathbb{Z}_2$-grading for the vector space V, so that $V = V_0 \oplus V_1$. Moreover, such a decomposition is orthogonal. Indeed, given $x \in V_0$ and $y \in V_1$, we have $xy + yx = 2g(x, y)$. As the left-hand side belongs to $\mathcal{C}\ell_1$ and the right-hand side to $\mathcal{C}\ell_0$, we must have $g(x, y) = 0$. Thus $V_0 \perp V_1$ and, in particular, $V_0 \subseteq V_1{}^\perp$. By counting dimensions, we finally have $V_0 = V_1{}^\perp$ (for $\dim(V_0) + \dim(V_1) = n$ and $\dim(V_1) + \dim(V_1{}^\perp) = n$). □

[5] In the following, we make the usual identification of V with $\Lambda^1(V)$ (as mentioned in section 1.1). In particular, items (ii) and (iv) of proposition 2 can be written, in a more precise way, as $\pi_i(\Lambda^1(V)) \subseteq \Lambda^1(V)$ and $\alpha(\Lambda^1(V)) \subseteq \Lambda^1(V)$, respectively.

Let the metric g have signature (p, q), with $p+q=n$, and let us denote the vector space V, endowed with g, by $\mathbb{R}^{p,q}$ (as in section 1.1). By proposition 4, we can choose orthonormal basis $\mathcal{B}_0=\{v_1,\ldots,v_a\}$ and $\mathcal{B}_1=\{v_{a+1},\ldots,v_{a+b}\}$ of V_0 and V_1 respectively. Then, $\{v_1,\ldots,v_{a+b}\}$ is an orthonormal basis for $\mathbb{R}^{p,q}$ and thus $\left\{1, v_{i_1}\cdots v_{i_k} : 1\leqslant i_1\leqslant\cdots\leqslant i_k\leqslant n, k=1,\ldots,n\right\}$ is an orthonormal basis for $\mathcal{C}\ell_{p,q}(\mathbb{R})$. It follows from (3) that $\mathcal{C}\ell_0$ is generated (as an algebra) by elements of the form

$$\text{(i) } v_i \qquad \text{with} \quad i\leqslant a \tag{4a}$$

$$\text{(ii) } v_i v_j \qquad \text{with} \quad i, j > a. \tag{4b}$$

This leads to a straightforward characterization of $\mathcal{C}\ell_0$. Let p_i (q_i) be the number of elements in $\mathcal{B}_i$ squaring to +1 (-1). As we work within $\mathcal{C}\ell_{p,q}(\mathbb{R})$, we have $p=p_0+p_1$ and $q=q_0+q_1$. Then, equations (4) imply that $\mathcal{C}\ell_0\cong\mathcal{C}\ell_{p_0,q_0}\otimes\mathcal{C}\ell^+_{p_1,q_1}$, where the tensor product is over $\mathbb{R}$ and comes from the fact that the elements in (i) and (ii) commute. We summarize this result in the following proposition.

Proposition 5. *Let $\mathcal{C}\ell_{p,q}(\mathbb{R})=\mathcal{C}\ell_0\oplus\mathcal{C}\ell_1$ be a $\mathbb{Z}_2$-grading preserving the multivector structure of $\Lambda(\mathbb{R}^{p,q})$. Then*

$$\mathcal{C}\ell_0\cong\mathcal{C}\ell_{p_0,q_0}\otimes\mathcal{C}\ell^+_{p-p_0,q-q_0} \tag{5}$$

where p_0 (q_0) is the number of α-even elements of an orthonormal basis of $\mathbb{R}^{p,q}$ squaring to +1 (-1).

Note the following:

(i) If $p_0=q_0=0$, then equation (5) reduces to $\mathcal{C}\ell_0=\mathcal{C}\ell^+_{p,q}(\mathbb{R})$, as expected. We refer to this case as the *usual* $\mathbb{Z}_2$-grading of $\mathcal{C}\ell_{p,q}(\mathbb{R})$.

(ii) If $p_0=p$ and $q_0=q$, then $\mathcal{C}\ell_0=\mathcal{C}\ell_{p,q}(\mathbb{R})$ and $\mathcal{C}\ell_1=0$. Therefore, this case corresponds to the *trivial* $\mathbb{Z}_2$-grading of $\mathcal{C}\ell_{p,q}(\mathbb{R})$. Moreover, this is the unique choice for p_0 and q_0 which yields the trivial $\mathbb{Z}_2$-grading.

It follows that every non-trivial $\mathbb{Z}_2$-grading, preserving the multivector structure of $\Lambda(V)$, provides an invertible α-odd element u (for example, any basis 1-vector in V_1 squaring to ±1 can be chosen for u). This can be used to construct the isomorphism of vector spaces $\mathcal{C}\ell_0\to\mathcal{C}\ell_1, x\mapsto ux$. Thus, the class of $\mathbb{Z}_2$-gradings considered here is such that either

$$\text{(a) } \dim\mathcal{C}\ell_0=\dim\mathcal{C}\ell_{p,q}(\mathbb{R}) \qquad \text{(trivial case)} \tag{6a}$$

or

$$\text{(b) } \dim\mathcal{C}\ell_0=\tfrac{1}{2}\dim\mathcal{C}\ell_{p,q}(\mathbb{R}). \tag{6b}$$

At this point, it is interesting to consider some examples of $\mathbb{Z}_2$-gradings that do not preserve the multivector structure of $\Lambda(V)$. For simplicity, let us momentarily regard the real Clifford algebra $\mathcal{C}\ell(V, g)$ as an algebra of $m\times m$ matrices, as in table 1. Let us define $\mathcal{C}\ell_0$ and $\mathcal{C}\ell_1$ as, respectively, the spaces of matrices of the form

$$\begin{pmatrix} A & 0_{a\times b}\\ 0_{b\times a} & B\end{pmatrix} \qquad \text{and} \qquad \begin{pmatrix} 0_{a\times a} & C\\ D & 0_{b\times b}\end{pmatrix}$$

where A and B are square matrices of order a and b, respectively, with $a+b=m$. It is easy to see that this gives a $\mathbb{Z}_2$-grading for $\mathcal{C}\ell(V, g)$ for any choice of a and b. In particular, if $m>2$, we can choose nonzero a and b such that $a\neq b$. As a result, we end up with a non-trivial $\mathbb{Z}_2$-grading with $\dim\mathcal{C}\ell_0\neq\frac{1}{2}\dim\mathcal{C}\ell_{p,q}(\mathbb{R})$. It follows from equations (6) that such $\mathbb{Z}_2$-grading does not preserve the multivector structure of $\Lambda(V)$.

Table 4. Even subalgebras ($\mathcal{C}\ell_0$) associated with $\mathbb{Z}_2$-gradings preserving the multivector structure of $\Lambda(\mathbb{R}^{p,q})$. The table exhibits $\mathcal{D}$ in $\mathcal{C}\ell_0 \cong \mathcal{M}(k,\mathbb{R}) \otimes \mathcal{D}$, where $k^2 \dim_{\mathbb{R}} \mathcal{D} = 2^{n-1}$ and $n = p+q$. Here, $p-q$ and $p_0 - q_0$ should be considered mod 8 and mod 4 respectively.

	$p-q$							
$p_0 - q_0$	0	1	2	3	4	5	6	7
0	$\mathbb{R}\oplus\mathbb{R}$	$\mathbb{R}$	$\mathbb{C}$	$\mathbb{H}$	$\mathbb{H}\oplus\mathbb{H}$	$\mathbb{H}$	$\mathbb{C}$	$\mathbb{R}$
1	$\mathbb{R}\oplus\mathbb{R}$	$\mathbb{R}\oplus\mathbb{R}\oplus\mathbb{R}\oplus\mathbb{R}$	$\mathbb{R}\oplus\mathbb{R}$	$\mathbb{C}\oplus\mathbb{C}$	$\mathbb{H}\oplus\mathbb{H}$	$\mathbb{H}\oplus\mathbb{H}\oplus\mathbb{H}\oplus\mathbb{H}$	$\mathbb{H}\oplus\mathbb{H}$	$\mathbb{C}\oplus\mathbb{C}$
2	$\mathbb{C}$	$\mathbb{R}$	$\mathbb{R}\oplus\mathbb{R}$	$\mathbb{R}$	$\mathbb{C}$	$\mathbb{H}$	$\mathbb{H}\oplus\mathbb{H}$	$\mathbb{H}$
3	$\mathbb{C}$	$\mathbb{C}\oplus\mathbb{C}$	$\mathbb{C}$	$\mathbb{C}\oplus\mathbb{C}$	$\mathbb{C}$	$\mathbb{C}\oplus\mathbb{C}$	$\mathbb{C}$	$\mathbb{C}\oplus\mathbb{C}$

Let us now return to the study of the $\mathbb{Z}_2$-gradings preserving the multivector structure of $\Lambda(\mathbb{R}^{p,q})$. The explicit formula for $\mathcal{C}\ell_0$ in proposition 5 can be used to obtain a complete classification for these algebras. In fact, a straightforward calculation (using the facts that $\mathbb{C} \otimes \mathbb{C} \cong \mathbb{C} \oplus \mathbb{C}, \mathbb{C} \otimes \mathbb{H} \cong \mathbb{C} \otimes \mathcal{M}(2,\mathbb{R})$ and $\mathbb{H} \otimes \mathbb{H} \cong \mathcal{M}(4,\mathbb{R})$) shows that $\mathcal{C}\ell_0 \cong \mathcal{M}(k,\mathbb{R}) \otimes \mathcal{D}$, where $\mathcal{D}$ is given by table 4 and k is fixed by $k^2 \dim_{\mathbb{R}} \mathcal{D} = 2^{n-1}$, where $n = p+q$. It is interesting to note that this yields an overall 4-fold periodicity in terms of $p_0 - q_0$.

We see from table 4 that $\mathcal{C}\ell_0$ is not always a Clifford algebra. For instance, when $p-q = 1 \pmod 4$ and $p_0 - q_0 = 1 \pmod 4$, we have $\mathcal{C}\ell_0 \cong \mathcal{M}(k,\mathbb{R}) \otimes (\mathbb{R}\oplus\mathbb{R}\oplus\mathbb{R}\oplus\mathbb{R})$, and we know that no Clifford algebra has this form.

3. Applications

Now we consider some simple applications of the framework developed in the previous section. In section 3.1, we outline a possible use of the $\mathbb{Z}_2$-gradings studied here to define spinor spaces, as in [5, 6]. In section 3.2, we analyse an algebraic method for changing the signature of arbitrary real Clifford algebras, as advanced in the introduction.

3.1. Spinor spaces

The identification of the even part of a Clifford algebra with a space of spinors is mostly known in the context of Hestenes' formulation of Dirac theory [2–4]. In such an approach, the state of the electron is described by an operator spinor [12] $\Psi \in \mathcal{C}\ell^+_{1,3}(\mathbb{R})$ satisfying the so called Dirac–Hestenes equation, $\partial\Psi e_{21} = m\Psi e_0$ (here $\{e_\mu\}$ is an orthonormal frame in Minkowski space, corresponding to a given observer, and $\partial = e^\mu \partial_\mu$). We observe that the space of operator spinors is more than a vector space, it is an *algebra*. This leads, among other things, to an elegant canonical decomposition for Ψ, which generalizes the polar decomposition of complex numbers. The Dirac–Hestenes equation is covariant under a change of frame/observer, for another choice $\{e'_\mu\}$ must be related to the old one by $e'_\mu = U e_\mu \tilde{U}$, with $U \in \mathrm{Spin}_+(1,3)$, yielding $\partial\Psi' e'_{21} = m\Psi' e'_0$, where $\Psi' = \Psi\tilde{U}$. On the other hand, the usual (matrix) Dirac equation is known to be covariant under a larger class of transformations, in which the gamma matrices γ_μ are transformed by $\mathsf{S}\gamma_\mu\mathsf{S}^{-1}$, where S is an arbitrary unitary matrix (this amounts to a change in the gamma matrix representation).

By considering this kind of transformation, it is possible to derive multivector Dirac equations associated with a large class of gamma matrix representations, including the standard, Majorana and chiral ones [6]. The resulting spinor spaces can be identified with even subalgebras $\mathcal{C}\ell_0$ of the kind considered in the previous section. Indeed, the generalized Dirac–Hestenes equation in this context reads $\breve{\partial}\Psi\sigma + m\Psi u = 0$, where $\Psi \in \mathcal{C}\ell_0$, σ and u are any commuting α-even and α-odd elements, respectively, satisfying $\sigma^2 = -1$ and $u^2 = 1$, and $\breve{\partial}\Psi := \pi_0(\partial)\Psi u + \pi_1(\partial)\Psi$ (see [6] for details). It follows that the resulting operator spinor spaces for the Dirac theory are isomorphic to either $\mathcal{M}(2,\mathbb{C})$ or $\mathbb{H}\oplus\mathbb{H}$. This method gives rise to a generalized spinor map, relating algebraic and operator spinors, which was used by us [6] to rederive certain quaternionic models of (the usual) quantum mechanics and to provide a natural way to obtain gamma matrix representations in terms of the enhanced $\mathbb{H}$-general linear group $GL(2,\mathbb{H})\cdot\mathbb{H}^*$ [13].

Let us now briefly consider more general Clifford algebras than $\mathcal{C}\ell_{1,3}(\mathbb{R})$. As was shown by Dimakis, it is always possible to represent a given $\mathcal{C}\ell_{p,q}(\mathbb{R})$ in itself, with a corresponding spinor space isomorphic to a subalgebra of the original algebra. This is done in [5], where such a subalgebra is obtained by taking the even part of successive $\mathbb{Z}_2$-gradings of $\mathcal{C}\ell_{p,q}(\mathbb{R})$. Moreover, this subalgebra is a real Clifford algebra by itself. Let us now outline a slight generalization of this procedure, in which the corresponding $\mathbb{Z}_2$-gradings are given as in the previous section. As we have seen, the resulting even subalgebra is not necessarily a real Clifford algebra in this case.

First of all, we note that the even subalgebra $\mathcal{C}\ell_0$ is in general too large to be taken as the space of spinors, which is classically given by a minimal one-sided ideal $\mathcal{I}$ in $\mathcal{C}\ell_{p,q}(\mathbb{R})$ [14]. Indeed, we have shown in the previous section that, for the (non-trivial) $\mathbb{Z}_2$-gradings considered here, $\dim\mathcal{C}\ell_0 = \frac{1}{2}\dim\mathcal{C}\ell_{p,q}(\mathbb{R})$. Thus, $\mathcal{I}$ and $\mathcal{C}\ell_0$ have the same dimension only for Clifford algebras isomorphic to 2×2 matrices, i.e., for $\mathcal{C}\ell_{2,0}(\mathbb{R})\cong\mathcal{M}(2,\mathbb{R})$, $\mathcal{C}\ell_{3,0}(\mathbb{R})\cong\mathcal{M}(2,\mathbb{C})$ and $\mathcal{C}\ell_{1,3}(\mathbb{R})\cong\mathcal{M}(2,\mathbb{H})$ (modulo isomorphisms) (see also section 10.8 of [7]). As we have already mentioned, the case $\mathcal{C}\ell_{1,3}(\mathbb{R})$ was analysed in [6]. On the other hand, for the Clifford algebra $\mathcal{C}\ell_{3,0}(\mathbb{R})$, which is related to Pauli theory in the same way as $\mathcal{C}\ell_{1,3}(\mathbb{R})$ is related to Dirac theory, our method leads to spinor spaces isomorphic to $\mathbb{H}$, $\mathcal{M}(2,\mathbb{R})$ or $\mathbb{C}\oplus\mathbb{C}$ (see table 4). Note that, as $\mathbb{C}\oplus\mathbb{C}$ is not a real Clifford algebra, this case is not given by Dimakis' method. A study of the Pauli equation along the lines of [6] would then result in three different (i.e. non-isomorphic) corresponding spinor algebras for this case.

For higher dimensional Clifford algebras, we can successively take $\mathcal{C}\ell_0, \mathcal{C}\ell_{00} = (\mathcal{C}\ell_0)_0$ and so on, having in mind that the prescription given by equation (5) works only when we have a real Clifford algebra involved. In the other cases, one might consider further generalizations of equation (5), such as $\mathcal{C}\ell^+_{p_0,q_0}\otimes\mathcal{C}\ell^+_{p-p_0,q-q_0}$ or $\mathcal{C}\ell_{p_0,q_0}\otimes\mathcal{C}\ell^{++}_{p-p_0,q-q_0}$.

3.2. Signature change in Clifford algebras

Let us now associate with each $\mathbb{Z}_2$-grading $\mathcal{C}\ell(V,g) = \mathcal{C}\ell_0\oplus\mathcal{C}\ell_1$, with corresponding grading automorphism α, the linear map $\Gamma_\alpha : V \to \mathrm{End}(V^\wedge)$ given by (cf equation (2))

$$\Gamma_\alpha(v) = v\wedge + \alpha(v)\lrcorner.$$

Proposition 6. *If the above $\mathbb{Z}_2$-grading preserves the multivector structure of $\Lambda(V)$, then Γ_α is a Clifford map for the pair (V, g_α), where given $u, v\in V$, the deformed metric g_α is defined by $g_\alpha(u,v) = g(u_0,v_0) - g(u_1,v_1)$, with $u_i = \pi_i(u)$, $v_i = \pi_i(v)$, $i = 0, 1$ (see footnote 5).*

Proof. As the $\mathbb{Z}_2$-grading is assumed to preserve the multivector structure of $\Lambda(V)$, we have $\alpha(v)\in V$, $\forall v\in V$. This yields $(\Gamma_\alpha(v))^2(x) = (v\wedge + \alpha(v)\lrcorner)(v\wedge x + \alpha(v)\lrcorner x) = v\wedge(\alpha(v)\lrcorner x) + \alpha(v)\lrcorner(v\wedge x) = (\alpha(v)\lrcorner v)x$, $\forall x\in V$. Therefore, $(\Gamma_\alpha(v))^2 = g(\alpha(v), v)1_{\Lambda(V)}$ and thus

$$(\Gamma_\alpha(v))^2 = \begin{cases} g(v,v)1_{\Lambda(V)} & \text{if} \quad v\in V_0 \\ -g(v,v)1_{\Lambda(V)} & \text{if} \quad v\in V_1 \end{cases}$$

where $V_i := V\cap\mathcal{C}\ell_i$, $i = 0, 1$ (as in the previous section). □

Under the conditions above, we can define a Clifford product $\vee_\alpha$ in $V^\wedge$, associated with Γ_α, by

$$v \vee_\alpha a = v \wedge a + \alpha(v) \lrcorner a \qquad v \in V \quad a \in V^\wedge$$

extended by linearity and associativity to all of $V^\wedge$. It follows that $(V^\wedge, \vee_\alpha)$ is the Clifford algebra associated with (V, g_α), where g_α is defined in proposition 6.

Given $v \in V$ and $a \in \Lambda^k(V)$, with $v_i := \pi_i(v)$, this product is related to the original Clifford product (denoted by juxtaposition) by $v \vee_\alpha a = v_0 \wedge a + v_0 \lrcorner a + v_1 \wedge a - v_1 \lrcorner a = v_0 a + (-1)^k (a \wedge v_1 + a \llcorner v_1) = v_0 a + \hat{a} v_1$. Therefore, the signature-changed product $\vee_\alpha$ may be written in terms of the original one as

$$v \vee_\alpha a = v_0 a + \hat{a} v_1 \tag{7}$$

where $v \in V$ and $a \in V^\wedge$. A more general expression for the $\vee_\alpha$-product between arbitrary multivectors may be obtained from the above formula by recursion.

Consider now the situation where one wants to change the metric signature from (p, q) to (r, s), with $p + q = r + s$ (see the introduction for a discussion on the instances where this can be useful). To accomplish that, we emulate the Clifford product associated with this new metric inside the algebraic structure of $\mathcal{C}\ell_{p,q}$, i.e., using only the algebraic data of $\mathcal{C}\ell_{p,q}$. More specifically, suppose that the square of some basis vectors $e_{i_1}, \ldots, e_{i_k}$, of an orthonormal basis $\{e_1, \ldots, e_n\} \in \mathbb{R}^{p,q}$, is required to change sign in this new setting, i.e., when viewed inside the signature-changed space. We then define a suitable $\mathbb{Z}_2$-grading for $\mathcal{C}\ell_{p,q}$ by declaring $e_{i_1}, \ldots, e_{i_k}$ as α-odd and the remaining basis vectors as α-even. In other words, we choose the α-parity of the elements in $\{e_1, \ldots, e_n\}$ by

	$\mathcal{C}\ell_0$	$\mathcal{C}\ell_1$
1-vectors	remaining e_i	$e_{i_1}, \ldots, e_{i_k}$

and let this choice generate the $\mathbb{Z}_2$-grading in which α-even (α-odd) elements are products of

(a) an even (odd) number of elements in $\{e_{i_1}, \ldots, e_{i_k}\}$;
(b) any number of elements in $\{e_1, \ldots, e_n\} \setminus \{e_{i_1}, \ldots, e_{i_k}\}$.

By the above proposition, the corresponding $\vee_\alpha$-product clearly implements the desired signature change $\mathcal{C}\ell_{p,q} \to \mathcal{C}\ell_{r,s}$. To clarify what is going on, we observe that we initially have a space of multivectors $V^\wedge = \bigoplus_{k=0}^n \Lambda^k(V)$ (which is not an algebra). Then, various products can be defined on $V^\wedge$. As we have seen, endowing $V^\wedge$ with the exterior product leads to the Grassmann algebra $\Lambda(V) = (V^\wedge, \wedge)$, while endowing $V^\wedge$ with the Clifford product leads to the Clifford algebra $\mathcal{C}\ell_{p,q} = (V^\wedge$, Clifford product). In the same way, the above arguments show that the Clifford algebra associated with the signature-changed metric (with signature (r, s)) is given by $\mathcal{C}\ell_{r,s} = (V^\wedge, \vee_\alpha)$. Moreover, the $\vee_\alpha$-product is parametrized by $\mathbb{Z}_2$-gradings and is related to the original Clifford product by equation (7).

Some examples are in order:

(i) For the *trivial* $\mathbb{Z}_2$-grading, where $\mathcal{C}\ell_0 = \mathcal{C}\ell_{p,q}$, i.e.,

	$\mathcal{C}\ell_0$	$\mathcal{C}\ell_1$
1-vectors	$e_1, \ldots, e_n$	—

we have $\alpha = \mathrm{id}_{\mathcal{C}\ell_{p,q}(\mathbb{R})}$ and thus $\vee_\alpha =$ [original product]. In other words, the trivial $\mathbb{Z}_2$-grading yields the trivial signature change (none).

(ii) For the *usual* $\mathbb{Z}_2$-grading, i.e.,

	$\mathcal{C}\ell_0$	$\mathcal{C}\ell_1$
1-vectors	—	$e_{i_1}, \ldots, e_n$

we have $\alpha|_{\mathbb{R}^{p,q}} = -\mathrm{id}_{\mathbb{R}^{p,q}}$ and thus $\vee_\alpha$ yields a change to the opposite metric $\mathcal{C}\ell_{p,q} \to \mathcal{C}\ell_{q,p}$. A straightforward calculation shows that given $a, b \in \mathcal{C}\ell_{p,q}$, we have $a \vee_\alpha b = b_0a_0 + b_0a_1 + b_1a_0 - b_1a_1$, where $a_i = \pi_i(a)$ and $b_j = \pi_j(b)$. This is precisely the tilt transformation introduced by Lounesto in [7].

(iii) For the $\mathbb{Z}_2$-grading

	$\mathcal{C}\ell_0$	$\mathcal{C}\ell_1$
1-vectors	$e_1, \ldots, e_{k-1}, e_{k+1}, \ldots, e_n$	e_k

we have $\mathcal{C}\ell_{p,q} \to \mathcal{C}\ell_{p-1,q+1}$ if e_k originally squares to +1 and $\mathcal{C}\ell_{p,q} \to \mathcal{C}\ell_{p+1,q-1}$ if e_k originally squares to -1.

(iv) Finally, for the arbitrary $\mathbb{Z}_2$-grading

	$\mathcal{C}\ell_0$	$\mathcal{C}\ell_1$
1-vectors	remaining e_i	$e_{i_1}, \ldots, e_{i_{\lvert r-p\rvert}}$

we have an arbitrary signature change $\mathcal{C}\ell_{p,q} \to \mathcal{C}\ell_{r,s}$. Therefore, the product $\vee_\alpha$ parametrizes all the possible signature changes in $\mathcal{C}\ell_{p,q}$ by means of $\mathbb{Z}_2$-gradings.

As a final remark, we note that Lounesto's tilt transformation can be alternatively generalized by the following prescription. Given $a_i \in \mathcal{C}\ell_i, b_j \in \mathcal{C}\ell_j$, we may define $a_i \vee'_\alpha b_j := (-1)^{ij} b_i a_j$, and extend $\vee'_\alpha$ as a bilinear product in $V^\wedge$. A straightforward calculation shows that $\vee'_\alpha$ is associative and preserves the $\mathbb{Z}_2$-graded structure of $\mathcal{C}\ell_{p,q}$ in question, in the sense that $\mathcal{C}\ell_i \vee'_\alpha \mathcal{C}\ell_j \subseteq \mathcal{C}\ell_{i+j\,(\mathrm{mod}\ 2)}$. By defining convenient $\mathbb{Z}_2$-gradings exactly as above, we see that $\vee'_\alpha$ also provides general signature change maps $\mathcal{C}\ell_{p,q} \to \mathcal{C}\ell_{r,s}$. However, the usual relation between the exterior product and the Clifford product must be accordingly changed. As a matter of fact, given two 1-vectors $x, y \in V$, we have $x \wedge y = \frac{1}{2}(xy - yx)$ but $x \wedge y = \sum_{ij} (-1)^{ij} \frac{1}{2}(y_i \vee'_\alpha x_j - x_j \vee'_\alpha y_i)$, where $x_i = \pi_i(x), y_j = \pi_j(y)$. In Lounesto's tilt to the opposite metric $\mathcal{C}\ell_{1,3} \to \mathcal{C}\ell_{3,1}$, the latter expression simplifies to $\frac{1}{2}(x \vee'_\alpha y - y \vee'_\alpha x)$, but it is easy to see that, in general, this is not the case.

4. Concluding remarks

We studied in detail an important class of $\mathbb{Z}_2$-graded structures on a real Clifford algebra $\mathcal{C}\ell(V, g)$. The corresponding $\mathbb{Z}_2$-gradings $\mathcal{C}\ell_0 \oplus \mathcal{C}\ell_1$ are required to preserve the multivector structure of the underlying Grassmann algebra over V (see definition 3). A complete classification for the associated even subalgebras, i.e., for $\mathcal{C}\ell_0$, was obtained. As preliminary applications, we first outlined the possibility of using such general even subalgebras as spinor spaces. After that, we employed such $\mathbb{Z}_2$-graded structures to deform the Clifford product of

$\mathcal{C}\ell(V, g)$, thereby parametrizing all the possible signature changes on this algebra. This can be useful in signature changing applications in theoretical physics (see the introduction). As we also mentioned in the introduction, the pervasiveness of $\mathbb{Z}_2$-graded structures in mathematical physics allows us to expect that yet other applications are likely to be found.

As a last remark, we would like to note that the opposite path to that considered here, with a fixed $\mathbb{Z}_2$-grading and alternative multivector structures, has been receiving considerable interest in the literature. Applications range from models in QFT [15] to q-quantization of Clifford algebras [16] (see also [11] and references therein).

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